

# A Simplified Algorithm for Augmenting Edge-Connectivity by One with Bipartition Constraints

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**Abstract**—The  $k$ -edge-connectivity augmentation problem with bipartition constraints ( $k$ ECABP, for short) is defined by “Given a multigraph  $G = (V, E)$  and a bipartition  $\pi = \{V_B, V_W\}$  of  $V$  with  $V_B \cap V_W = \emptyset$ , find an edge set  $E_f$  of minimum size, consisting of edges that connect  $V_B$  and  $V_W$ , such that  $G_f = (V, E \cup E_f)$  is  $k$ -edge-connected, where a multigraph means a graph, with unweighted edges, such that multiple edges may exist.” In this paper, we give a simplified algorithm for finding an optimal solution to  $(\sigma + 1)$ ECABP in  $O(|V||E| + |V|^2 \log |V|)$  time when  $G$  is  $\sigma$ -edge-connected ( $\sigma > 0$ ), and show that the problem can be solved in linear time when  $1 \leq \sigma \leq 2$ . The time complexity of the proposed algorithm is equal to that of an existing algorithm of Oki et al. (2012) (to appear).

## I. INTRODUCTION

**[Background]** The  $k$ -edge-connectivity augmentation problem ( $k$ ECA, for short) is defined by “Given a multigraph  $G = (V, E)$ , find an edge set  $E_f$  of minimum cardinality such that  $G_f = (V, E \cup E_f)$  is  $k$ -edge-connected, where a multigraph means a graph, with unweighted edges, such that multiple edges may exist.” We often denote  $G_f$  as  $G + E_f$ , and  $E_f$  is called an optimal solution to the problem. There are several applications to construction of a fault-tolerant network, and so on. It is called the  $k$ -edge-connectivity augmentation problem with bipartition constraints ( $k$ ECABP, for short) when a bipartition  $\pi = \{V_B, V_W\}$  of  $V$  with  $V_B \cap V_W = \emptyset$  is additionally given and we require that  $E_f$  consists of edges connecting between  $V_B$  and  $V_W$  (see Fig. 1).

A bipartite graph is a graph  $(V, E)$  such that  $V$  is partitioned into two sets  $V^B$  and  $V^W$  with  $V^B \cap V^W = \emptyset$ , and any edge  $(u, v) \in E$  satisfies a condition ( $u \in V^B$  and  $v \in V^W$ ) or ( $u \in V^W$  and  $v \in V^B$ ): such a graph is often denoted as  $G = (V^B \cup V^W, E)$ . If  $G$  is bipartite and we set  $V_B = V^B$  and  $V_W = V^W$  in  $k$ ECABP then  $G_f = G + E_f$  is bipartite.

This problem, denoted as B- $k$ ECABP, is a typical subproblem of  $k$ ECABP, where “B-” means that  $G$  is a bipartite graph. There are several applications to security of statistical data stored in a cross tabulated table [5], and so on.

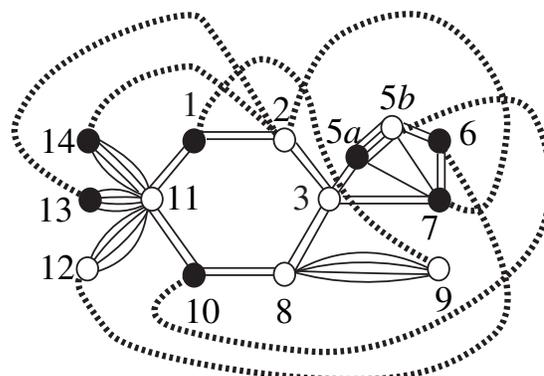


Fig. 1. A graph  $G$  with  $\lambda(G) = 4$ , where a closed circle (an open circle, respectively) represents a vertex which belongs to  $V_B$  ( $V_W$ ). The set of dashed lines is an optimal solution  $E_f = \{(1, 9), (2, 7), (2, 13), (2, 14), (5b, 10), (6, 12)\}$  to 5ECABP.

**[Existing Results]** Many algorithms for  $k$ ECA have been given. [3] gave a linear time algorithm for 2ECA, and [16], [4], [9] gave polynomial time algorithms for  $k$ ECA.

[5] gave a linear time algorithm for B-2ECABP, and an  $O(\log |V|)$  parallel time algorithm on an EREW PRAM with a linear number of processors. A general problem  $k$ ECAMP is similarly defined, where  $r$ -partition  $\pi_M = \{V_1, \dots, V_r\}$  ( $r \geq 2$ ) of  $V$  is given and  $E_f$  consists of edges connecting between  $V_i$  and  $V_j$  ( $1 \leq i < j \leq r$ ). Several algorithms for  $k$ ECAMP have been given: [1] gave an  $O(|V|(|E| + |V| \log |V|) \log |V|)$  time algorithm. Note that, in [1], a given multigraph is handled as an edge-weighted simple one such that, for any pair of vertices  $u$  and  $v$ , a simple edge  $(u, v)$  with a weight  $w(u, v) = x$  means that there are  $x$  multiple edges between  $u$  and  $v$ .

Let M- $k$ ECAMP denote  $k$ ECAMP in which  $G$  is an  $r$ -partite graph, where “M-” means that  $G$  is a multipartite graph. [2] gave a linear time algorithm for 2ECAMP, and an  $O(\log |V|)$  parallel time algorithm on an EREW PRAM with a linear number of processors. [12] gave an  $O(|V||E| + |V|^2 \log |V|)$  time

algorithm for  $(\sigma + 1)$ ECABP.

**[Our Contributions]** Now we describe our contributions of the paper as follows.

Our main contribution of the paper is to give a simplified algorithm  $\mathbf{S-Sol\_}(\sigma + 1)$ ECABP for finding an optimal solution to  $(\sigma + 1)$ ECABP in linear time when  $G$  is  $\sigma$ -edge-connected and a structural graph  $F(G)$  of  $G$  is given. Note that a structural graph  $F(G)$  represents all minimum cuts of  $G$ .

The time complexity of  $\mathbf{S-Sol\_}(\sigma + 1)$ ECABP is  $O(|V||E| + |V|^2 \log |V|)$  because  $F(G)$  can be constructed in  $O(|V||E| + |V|^2 \log |V|)$  time [10]. It follows that the problem can be solved in linear time when  $1 \leq \sigma \leq 2$  because  $F(G)$  can be obtained in  $O(|V| + |E|)$  time [8], [13].

The main theorem is as follows and its proof is given in § IV-D.

**Theorem 1.1** **Algorithm**  $\mathbf{S-Sol\_}(\sigma + 1)$ ECABP finds an optimal solution to  $(\sigma + 1)$ ECABP in  $O(|V||E| + |V|^2 \log |V|)$  time. Moreover, it runs in  $O(|V| + |E|)$  time when  $1 \leq \sigma \leq 2$ .

**[The Structure of the Paper]** The paper is organized as follows. Section II provides some definitions and notations. Section III shows a lower bound on the size of feasible solutions to  $(\sigma + 1)$ ECABP. Section IV gives formal description of  $\mathbf{S-Sol\_}(\sigma + 1)$ ECABP, and in Section IV-C, its correctness is provided, and in Section IV-D, its time complexity is provided. The concluding remarks are given in Section V.

## II. DEFINITIONS

In this section, we explain basic terminologies of graphs and structural graphs. Also mentioned is handling of a multigraph: unweighted multigraphs versus edge-weighted simple graphs. **[Basic Terminologies of Graphs]** An *undirected graph* is denoted as  $G = (V(G), E(G))$ , where  $V(G)$  and  $E(G)$  are often denoted as  $V$  and  $E$ , respectively. In this paper, only graphs without loops are handled, and the term “a graph” means an undirected multigraph unless otherwise stated. An edge that is incident to two vertices  $u, v$  in  $G$  is denoted as  $(u, v)$ . For two graphs  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$ ,  $H$  is a *subgraph* of  $G$  if  $V(G) \subseteq V(H)$  and  $E(G) \subseteq E(H)$ .

**Remark 2.1** It should be noted that conceptually a multigraph is used, while the corresponding edge-weighted simple graph is adopted in actual handling of graphs in this paper.

For a set  $X \subseteq V$  of  $G$ , let  $G[X]$  denote the subgraph having  $X$  as its vertex set and  $\{(u, v) \in E \mid u, v \in X\}$  as its edge set.

For two disjoint sets  $X, X' \subset V$ , we denote  $(X, X'; G) = \{(u, v) \in E \mid u \in X \text{ and } v \in X'\}$ , where it is often denoted as  $(X, X')$  if  $G$  is clear from the context. The *degree* of  $X$  (in  $G$ ) is defined by  $d_G(X) = |(X, V - X; G)|$ . If  $X = \{v\}$  then  $d_G(\{v\})$  is the total number of edges incident to  $v$  and is called the degree of  $v$  (in  $G$ ):  $d_G(\{v\})$  is denoted as  $d_G(v)$ . For a set  $X \subseteq V$ , the set  $(X, V - X; G)$  is called a  $k$ -cut if  $|(X, V - X)| = k$ .

For an edge set  $E_f$  consisting of edges with endvertices in  $V$ , let  $G + E_f$  denote the graph  $(V, E \cup E_f)$ . For a given bipartition  $\pi = \{V_B, V_W\}$  of  $V$  with  $V_B \cap V_W = \emptyset$ ,  $E_f$  is said to be *legal* (with respect to  $\pi$ ) if  $E_f$  consists of edges connecting

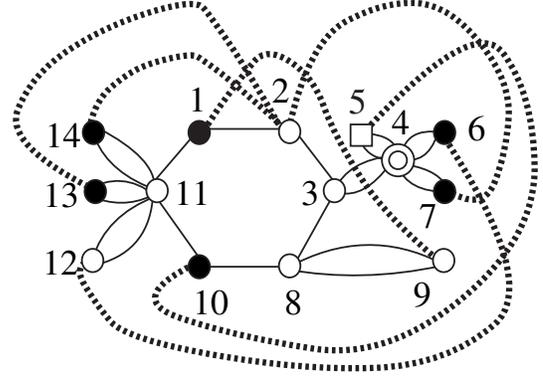


Fig. 2. A structural graph  $F(G)$ , where a closed circle (an open circle and a square, respectively) represents a vertex which belongs to  $BF(G)$  ( $WF(G)$  and  $HF(G)$ ). The vertex 4 (double circled) is inserted in constructing  $F(G)$ . The set of dashed lines denotes an optimal solution  $E' = \{(1, 9), (2, 7), (2, 13), (2, 14), (5, 10), (6, 12)\}$  to a structural graph  $F(G)$  of  $G$  in Fig. 1 and  $\lambda(F(G) + E') = \lambda(F(G)) + 1$ .

between different members  $V_B$  and  $V_W$  of  $\pi$ . The term “with respect to  $\pi$ ” is omitted if  $\pi$  is fixed.

A *trail* (or a  $(v_0, v_r)$ -trail) is a sequence of distinct edges  $(v_0, v_1), (v_1, v_2), \dots, (v_{r-1}, v_r)$  in which there may appear the same endvertices, and each of  $v_0$  and  $v_r$  is called a *terminal vertex* of the trail. A trail is called a *closed trail* if  $r \geq 1$  and  $v_0 = v_r$ . A closed trail is called an *Eulerian closed trail* of  $G$  if all edges of  $G$  are included. A trail is called a *path* (or a  $(v_0, v_r)$ -path) if all vertices  $v_0, v_1, \dots, v_r$  are distinct. A *cycle* consists of a path with  $r \geq 1$  and an edge  $(v_r, v_0)$ .

For two vertices  $u, v \in V$ , let  $\lambda(u, v; G)$  denote the maximum number of edge-disjoint  $(u, v)$ -paths between  $u$  and  $v$  in  $G$ . For a subset  $\Gamma \subseteq V$ , the *edge-connectivity* of  $\Gamma$  in  $G$  is defined by  $\lambda(\Gamma; G) = \min_{u, v \in \Gamma} \lambda(u, v; G)$ . The *edge-connectivity* of  $G$  is defined as  $\lambda(V; G)$ , and is denoted as  $\lambda(G)$ .  $G$  is  $k$ -*edge-connected* if  $\lambda(G) \geq k$  for a nonnegative integer  $k$ . In particular,  $G$  is *connected* if  $G$  is 1-edge-connected. For a set  $S \subseteq V$ ,  $S$  is a  $k$ -*edge-connected component* ( $k$ -component, for short) of  $G$  if  $S$  is a maximal vertex set such that  $\lambda(u, v; G) \geq k$  holds for any pair of vertices  $u, v \in S$ . For a  $k$ -component  $S$ ,  $S$  is a *leaf  $k$ -component* if  $d_G(S) = \lambda(G)$ . Note that distinct  $k$ -components are pairwise disjoint. For a  $k$ -cut  $(X, V - X; G)$ ,  $(X, V - X; G)$  is a minimum cut if  $k = \lambda(G)$ . A vertex  $v$  is called a *cutvertex* of  $G$  if the number of 1-components of  $G - v$  is greater than that of  $G$ , where  $G - v$  denotes  $G[V - \{v\}]$ . In this paper, let  $\sigma = \lambda(G)$ .

A *tree* is a connected graph which does not contain any cycle as its subgraph. A *cactus* is an undirected connected graph in which any pair of cycles shares at most one vertex: any vertex  $v$  with  $1 \leq d_G(v) \leq 2$  is called a *leaf* and any shared vertex is a cutvertex.

**[Structural Graphs]** A structural graph  $F(G) = (V(F(G)), E(F(G)))$  (see [10], [6] for example) of  $G$  with  $\lambda(G) = \sigma$  is a representation of all minimum cuts of  $G$  (see Fig. 2).  $F(G)$  is an edge-weighted cactus of  $O(|V|)$  vertices and edges such that each tree edge (a bridge in  $F(G)$ ) has weight  $\lambda(G)$  and each cycle edge (an edge included in a cycle) has weight  $\lambda(G)/2$ .

Particularly if  $\lambda(G)$  is odd then  $F(G)$  is an edge-weighted tree.

Any minimum cut ( $\sigma$ -cut) of  $G$  corresponds to either a tree edge or a pair of two cycle edges in the same cycle of  $F(G)$ , and vice versa. Each  $(\sigma + 1)$ -component of  $G$  is represented as a vertex of  $F(G)$ . In particular each leaf  $(\sigma + 1)$ -component of  $G$  corresponds to a leaf of  $F(G)$ , and vice versa. Several vertices, called *empty vertices*, are added in order to form a cactus.

Let  $\rho: V(G) \rightarrow V(F(G))$  denote such a mapping. We use the following notations:  $\rho(X) = \{\rho(v) \mid v \in X\}$  for  $X \subseteq V$  and  $\rho^{-1}(Y) = \{v \in V \mid \rho(v) \in Y\}$  for  $Y \subseteq V(F(G))$ . If  $Y = \{v\}$  then  $\rho^{-1}(Y)$  is written as  $\rho^{-1}(v)$ , instead of  $\rho^{-1}(\{v\})$ .

We can regard  $F(G)$  as a *modified cactus* which is defined as follows: if  $F(G)$  has any bridge of weight  $\lambda(G)$  then we replace such a bridge by a pair of multiple edges, assigning each edge weight  $\lambda(G)/2$  even if  $\lambda(G)$  is odd, and regard such a pair of multiple edges as a simple cycle of length two. We assume that  $F(G)$  is a modified cactus in this paper unless otherwise stated.  $F(G)$  can be obtained in  $O(|V||E| + |V|^2 \log |V|)$  time as a modified cactus [10].

Note that we can handle this modified cactus  $F(G)$  as a structural graph of  $G$  and  $\lambda(F(G)) = 2$ . Even if  $\lambda(G)$  is odd, finding an edge set  $E_f$  such that  $\lambda(G + E_f) = \lambda(G) + 1$  can be solved by finding an edge set  $E'$  such that  $\lambda(F(G) + E') = 3$  for a modified cactus  $F(G)$ . This is because there is a bijection  $\xi: E' \rightarrow E_f$  such that  $\xi((u, v)) = (n_u, n_v)$ , with  $\rho(n_u) = u$  and  $\rho(n_v) = v$ .

A vertex  $y \in V(F(G))$  with  $\rho^{-1}(y) = \emptyset$  is an empty vertex. Let  $\varepsilon(G) \subseteq V(F(G))$  denote the set of all empty vertices of  $F(G)$ . See [10] for efficiently constructing  $F(G)$  from  $G$ .

Let us call each vertex of  $V_B$  (of  $V_W$ , respectively) a black vertex (a white one) of  $G$ . Given a structural graph  $G = (V, E)$  with a bipartition  $\pi = \{V_B, V_W\}$ , we classify vertices  $v \in V(F(G)) - \varepsilon(G)$  into three types as follows:

- (i)  $\rho^{-1}(v) \subseteq V_B$  ( $\rho^{-1}(v) \subseteq V_W$ , respectively) ( $v$  is called a *black vertex* (a *white one*) of  $F(G)$ );
- (ii)  $\rho^{-1}(v) \cap V_B \neq \emptyset$  and  $\rho^{-1}(v) \cap V_W \neq \emptyset$  ( $v$  is called a *hybrid one* of  $F(G)$ ).

The set of *black leaves* (*white ones*, or *hybrid ones*, respectively) of  $F(G)$  is denoted as  $BF(G)$  ( $WF(G)$ , or  $HF(G)$ ). Let  $LF(G) = BF(G) \cup WF(G) \cup HF(G)$ . In this paper, without loss of generality, we assume that  $V_B \neq \emptyset$ ,  $V_W \neq \emptyset$  and  $|BF(G)| \geq |WF(G)|$ .

We say that  $F(G)$  is *B-dominant* [5] if  $|BF(G)| > |WF(G)| + |HF(G)|$  holds.

In figures for  $F(G)$  of this paper, we represent a hybrid vertex as a square, and a black one or a white one as a closed circle or an open one (see Fig. 2).

### III. A LOWER BOUND ON A FEASIBLE SOLUTION TO $(\sigma + 1)$ ECABP

Since  $(\sigma + 1)$ ECABP is a subproblem of  $k$ ECAMP, we obtain the following proposition by setting  $k = \sigma + 1$  for a lower bound shown in [1] on  $k$ ECAMP.

Assume that  $F(G)$  has a sequence of  $t$  pairs ( $t \geq 1$ ) of multiple edges (consisting of a simple cycle of length two) from a vertex  $u_0$  to a vertex  $u_t$ . Then let us shrink all

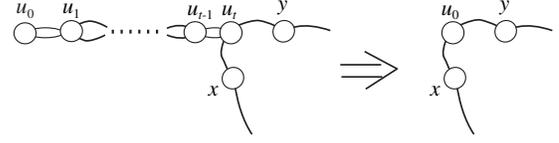


Fig. 3. Schematic explanation of 2-cycle-pruning from a leaf  $u_0$  to a vertex  $u_t$  ( $t \geq 1$ ) of  $F(G)$  in a cycle of length at least three of  $F(G)$ .

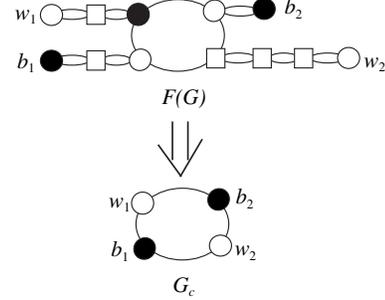


Fig. 4. An example of constructing  $G_c$  from a given  $F(G)$  (by repeating 2-cycle-pruning).

vertices  $u_0, \dots, u_{t-1}$  to  $u_t$  and remove any resulting self-loop, and then rename  $u_t$  as  $u_0$ . If  $u_0$  is a black or a white or a hybrid vertex then so is the renamed vertex  $u_0$ . We call this operation *2-cycle-pruning* (from  $u_0$  to  $u_t$ ). Fig. 3 shows schematic explanation of 2-cycle-pruning when  $u_t$  is included in a cycle of length at least three. If  $x = y$  in Fig. 3 then 2-cycle-pruning can be repeated.

**Proposition 3.1** [12] Let  $G_c$  be any graph constructed from  $F(G)$  by repeating 2-cycle-pruning as many times as possible. A lower bound  $\mathcal{L}$  on the number of edges required in augmenting edge-connectivity of  $G$  by one bipartition is given as follows:

- (i) If  $|LF(G)| = 4$  and  $G_c$  is a simple cycle of length four such that two black leaves and two other ones (white one or hybrid one) appear alternately then  $\mathcal{L} = 3$ ;
- (ii) Otherwise,  

$$\mathcal{L} = \max\{|BF(G)|, |WF(G)|, \lceil |LF(G)|/2 \rceil\}.$$

In the next section, we show that **Algorithm S-Sol** $_{(\sigma + 1)$ ECABP finds an edge set whose size is equal to the lower bound of Proposition 3.1, giving us an optimal solution  $E_f$ .

### IV. A SIMPLIFIED ALGORITHM FOR $(\sigma + 1)$ ECABP

In this section, we give a simplified algorithm **S-Sol** $_{(\sigma + 1)$ ECABP for  $(\sigma + 1)$ ECABP.

#### A. Ideas for the Algorithm

Now we show ideas for designing the algorithm and the main procedure.

**[A Linear Ordering on a Structural Graph]** Now we explain a *linear ordering* on vertices of a modified cactus  $F(G)$ . This is introduced in [11] for finding an optimal solution to 3ECA for  $F(G)$  efficiently, and is obtained as follows.

First all simple cycles in the cactus are assigned distinct colors. Note that this “color” is different from a color “black” or “white” used to present bipartition constraints. This coloring can be done in  $O(|V| + |E|)$  time based on a depth-first search.

Next another depth-first search starts at an arbitrary vertex according to the following manner: if any vertex  $u$  is visited for the first time via an edge included in some simple cycle (for example, its color is red) then, before traversing another edge which is in the red cycle and incident to  $u$ , the other edges incident to  $u$  are traversed.

This search assigns a linear ordering (denoted as  $\hat{\beta}(v)$ ) to each vertex  $v$  of  $V(F(G))$  from 1 to  $|V(F(G))|$ , and traversing vertices  $v$  in the order of  $\hat{\beta}(v)$  from 1 to  $|V(F(G))|$  makes an Eulerian closed trail  $ET(F(G))$  of  $F(G)$ . A vertex  $v \in V(F(G))$  appears more than once in  $ET(F(G))$  if and only if  $v$  is a cutvertex of  $F(G)$ .

Suppose that we fix a linear ordering  $\hat{\beta}(v)$  for  $v \in V(F(G))$ . By traversing  $ET(F(G))$  in the order of  $\hat{\beta}(v)$  for  $v \in V(F(G))$  from 1 to  $|V(F(G))|$  also assigns another linear ordering  $\beta(v)$  for leaves  $v \in LF(G)$ , that is, all leaves of  $F(G)$  are numbered according to  $\hat{\beta}$  by skipping cutvertices. Let us denote  $LF(G) = \{v_1, \dots, v_{|LF(G)|}\}$  with indices denoting this ordering  $\beta$ . Put  $t = \lceil |LF(G)|/2 \rceil$ . Let  $E_F$  be an edge set defined as follows:

$$E_F = \begin{cases} \{(v_i, v_{i+t}) \mid i = 1, \dots, t\} & \text{if } |LF(G)| \text{ is even} \\ \{(v_i, v_{i+t}) \mid i = 1, \dots, t-1\} \cup \{(v_t, v_1)\} & \text{if } |LF(G)| \text{ is odd} \end{cases}$$

Clearly  $|E_F| = \lceil |LF(G)|/2 \rceil$ , and it follows from the result of [11] that  $E_F$  is an optimal solution to 3ECA for a modified cactus  $F(G)$ .

**[An Optimal Solution and a Structural Graph]** We can focus on the case where  $F(G)$  is a modified cactus with  $\lambda(F(G)) = 2$ . From properties of a structural graph, it is enough to solve 3ECABP for  $F(G)$ , instead of  $(\sigma + 1)$ ECABP for  $G$  (see [11], [7], for example). We call an optimal solution  $E'$  to 3ECABP for  $F(G)$  simply as “an optimal solution  $E'$  to  $F(G)$ ”. Note that  $|LF(G + \xi(E'))| = 0$ .

It is easy to see that any optimal solution  $E'$  to  $F(G)$  requires the following (i) and (ii).

- (i) Edges  $(u, v) \in E'$  connect as many leaves as possible (in order to efficiently augment the edge-connectivity of  $G$  by one);
- (ii)  $n_u$  or  $n_v$ , respectively, should be a black vertex in  $\rho_G^{-1}(u)$  or a white one in  $\rho_G^{-1}(v)$  (in order to keep bipartition constraints).

**Algorithm S-Sol<sub>-</sub>( $\sigma + 1$ )ECABP** and **Procedure FindEdgesBP** are outlined as follows.

**[Algorithm S-Sol<sub>-</sub>( $\sigma + 1$ )ECABP]** We outline how to find an optimal solution  $E'$  to  $F(G)$ .

First, in order to narrow the gap between the number of black leaves and that of white ones, each hybrid leaf is appropriately regarded as a black leaf or a white one, since any hybrid leaf can be treated as a black or white one.

Next we find an edge set  $E'_p$  by **Procedure FindEdgesBP**, where the edge set  $E'_p$  consists of edges connecting different members of a given bipartition. As input to the procedure we

choose a leaf set  $L_p = W_p \cup B_p$ , where  $W_p$  consists of all white leaves and  $B_p$  is a set of black ones arbitrarily selected so that the number of black ones may be equal to that of white ones. Note that  $|L_p|$  is even. If  $F(G)$  is  $B$ -dominant then, after adding  $E'_p$  to  $F(G)$ , some black leaves (or possibly hybrid ones) are left. Thus we add edges, each connecting a black leaf and either a hybrid one or a white vertex which is not a leaf.

It follows that an optimal solution  $E'$  to  $F(G)$  is obtained and then we convert it into an optimal solution  $E_f$  to  $G$ .

**[Procedure FindEdgesBP]** When  $|LF(G)| \neq 4$ , it finds an edge set  $E'_p$  to be added to  $F(G)$  such that  $L_p$  is included of a 3-component  $S$  of  $F(G) + E'_p$  and  $E'_p$  is legal.

Let us linearly order vertices of  $L_p$  as  $\{v_1, \dots, v_{|L_p|}\}$  by traversing  $ET(F(G))$  according to a fixed linear ordering  $\beta$ . Let  $\beta_p(v)$  denote this new ordering on vertices  $v \in L_p$ , where if  $\beta(u) < \beta(v)$  then  $\beta_p(u) < \beta_p(v)$  for  $u, v \in L_p$ .

First, if  $|L_p| = 2$  then we can easily obtain such an edge set  $E'_p$  with  $|E'_p| = 1$ . Next we consider the case with  $|L_p| \geq 6$ .

There are two cases:

- (i) There exists a pair  $b \in B_p$  and  $w \in W_p$  satisfying either  $\beta_p(b) + |L_p|/2 = \beta_p(w)$  or  $\beta_p(b) = \beta_p(w) + |L_p|/2$  ( $w = v_1$  and  $b = v_{1+|L_p|/2}$  in Fig. 5);
- (ii) Otherwise (see Fig. 6).

Let  $L_{p_1} = \{v_2, \dots, v_{|L_p|/2}\}$  and  $L_{p_2} = \{v_{2+|L_p|/2}, \dots, v_{|L_p|}\}$  in Fig. 5, while  $L_{p_1} = \{v_3, \dots, v_{|L_p|/2}\}$  and  $L_{p_2} = \{v_{3+|L_p|/2}, \dots, v_{|L_p|}\}$  in Fig. 6.

[The case (i)] As shown in Fig. 5, the number of black leaves in  $L_{p_1}$  is equal to that of white leaves in  $L_{p_2}$  and the number of white leaves in  $L_{p_1}$  is equal to that of black leaves in  $L_{p_2}$ , because  $B_p \cup W_p = L_p$ ,  $|B_p| = |W_p|$  and  $|L_{p_1}| = |L_{p_2}|$ .

Then we can easily construct a matching  $\mathcal{M}$  consisting of edges  $(u, w)$  satisfying either  $(u \in W_p \cap L_{p_1}$  and  $w \in B_p \cap L_{p_2})$  or  $(u \in B_p \cap L_{p_1}$  and  $w \in W_p \cap L_{p_2})$ . Let  $E'_p = \mathcal{M} \cup \{(v_1, v_{1+|L_p|/2})\}$ .

[The case (ii)] First we find two leaves  $w \in W_p$  and  $b \in B_p$  with consecutive order of  $\beta_p$  (in Fig. 6,  $w = v_1$  and  $b = v_2$ ). In this case,  $u \in L_p$  with  $|\beta_p(w) - \beta_p(u)| = |L_p|/2$  is a white leaf and  $v \in L_p$  with  $|\beta_p(b) - \beta_p(v)| = |L_p|/2$  is a black leaf (in Fig. 6,  $u = v_{1+|L_p|/2} \in W_p$  and  $v = v_{2+|L_p|/2} \in B_p$ ). Similarly to (i),  $|L_{p_1} \cap B_p| = |L_{p_2} \cap W_p|$  and  $|L_{p_1} \cap W_p| = |L_{p_2} \cap B_p|$ . Hence we can construct a matching  $\mathcal{M}$  similarly to (i). Let  $E'_p = \mathcal{M} \cup \{(v_1, v_{2+|L_p|/2}), (v_2, v_{1+|L_p|/2})\}$ .

Let us note that **FindEdgesBP** is a combination of two procedures **ETC** and **AETC** in [5]. **ETC** and **AETC**, however, can be used only to 2-edge-connect a given connected graph, while we generalize them so that may be used in augmenting edge-connectivity by one for a  $\sigma$ -edge-connected graph ( $\sigma \geq 1$ ). **ETC** (**AETC**, respectively) corresponds to Step 6 (Step 11) of **FindEdgesBP**.

### B. Description of the Algorithm

We give formal description of **Algorithm S-Sol<sub>-</sub>( $\sigma + 1$ )ECABP** and **Procedure FindEdgesBP**.

**Algorithm S-Sol<sub>-</sub>( $\sigma + 1$ )ECABP**

**Input:** A connected graph  $G = (V, E)$ ,

with a bipartition  $\pi = \{V_B, V_W\}$  with  $|V_B| \geq |V_W|$ .

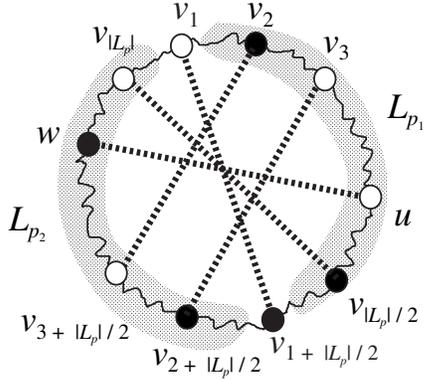


Fig. 5. Schematic explanation of  $E'_p$  at Step 6 of FindEdgesBP: each dashed line denotes an edge of  $E'_p$  and wavy lines show an Eulerian closed trail  $ET(F(G))$ .

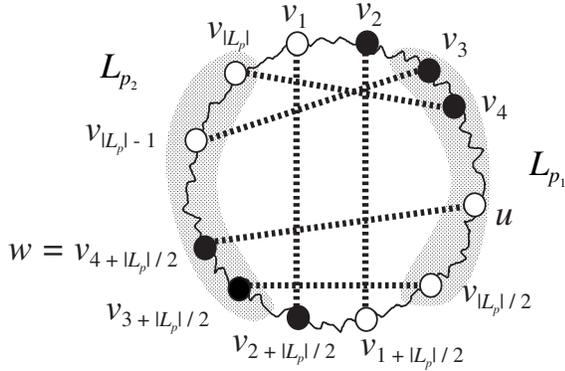


Fig. 6. Schematic explanation of  $E'_p$  at Step 11 of FindEdgesBP: each dashed line denotes an edge of  $E'_p$  and wavy lines show an Eulerian closed trail  $ET(F(G))$ .

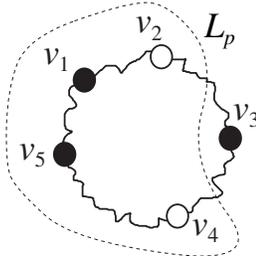


Fig. 7. Schematic explanation of Step 11 (i) of S-Sol $_{(\sigma+1)ECABP}$ : choosing  $\{v_5, v_1\}$  as  $B_p$  ensures existence of a pair  $v_a$  and  $v_{a+|L_p|/2}$  in Step 3 of FindEdgesBP, where wavy lines show an Eulerian closed trail  $ET(F(G))$ .

**Output:** An edge set  $E_f$  with minimum size such that  $(V, E \cup E_f)$  is  $(\sigma + 1)$ -edge-connected and  $E_f$  is legal.

- 1: Construct a structural graph  $F(G) = (V(F(G)), E(F(G)))$ ;
- 2: Find a linear ordering  $\beta$ ;
- 3:  $E'_2 \leftarrow \emptyset$ ,  $B \leftarrow BF(G)$ ,  $W \leftarrow WF(G)$ ,  $H \leftarrow HF(G)$ , and  $L \leftarrow LF(G)$ ;
- 4: **if**  $H \neq \emptyset$  **then**
- 5: Choose  $\min\{\lfloor (|L| - 2|W|)/2 \rfloor, |H|\}$  hybrid leaves, insert them into  $W$  (regarded as white ones) and insert the other hybrid ones into  $B$ ;
- 6:  $H \leftarrow \emptyset$ ; /\* After this step, we have  $|B| \geq |W|$  and  $|L| = |B| + |W|$ , since  $|W| \leq \lfloor |L|/2 \rfloor$ . \*/
- 7: **end if**
- 8: **if**  $|LF(G)| = 4$  **then**
- 9: Find an edge set  $E_f$  by Lemma 4.2, output  $E_f$  and terminate;
- 10: **else** /\*  $|LF(G)| \neq 4$  \*/
- 11: Choose a leaf set  $B_p \subseteq B$  with  $|B_p| = |W|$  as in (i) or (ii):
  - (i) If  $|W| = 2$  two black leaves  $b, b' \in B$  ( $|B| \geq 3$ ) satisfying either  $(\beta(b) + 1 = \beta(b'))$  or  $(\beta(b) = 1$  and  $|L| = \beta(b'))$  (see Fig. 7) and  $B_p \leftarrow \{b, b'\}$ ;
  - (ii) Otherwise /\*  $|W| \geq 3$  \*/ choose  $B_p \subseteq B$  arbitrarily;
- 12:  $W_p \leftarrow W$ ,  $L_p \leftarrow B_p \cup W_p$ ;
- 13: (i) Compute a linear ordering  $\beta_p$  on  $L_p$  by traversing  $ET(F(G))$  according to  $\beta$  such that, for any  $v, w \in L_p$ , if  $\beta(v) < \beta(w)$  then  $\beta_p(v) < \beta_p(w)$ ;
- (ii) Let  $L_p = \{v_1, \dots, v_{|L_p|}\}$  with indices denoting the order of  $\beta_p$ ;
- 14: Find an edge set  $E'_1$  by applying FindEdgesBP to  $L_p$ ;
- 15:  $B \leftarrow B - B_p$ ;
- 16: **end if**
- 17: **if**  $|B| > 0$  **then**
- 18:  $E'_2 \leftarrow \{(b, w) \mid b \in B\}$ , where  $w$  is set to a leaf  $v_1 \in WF(G) \cup HF(G)$  defined at Step 2, 5 or 10 of FindEdgesBP;
- 19: **end if**
- 20:  $E' \leftarrow E'_1 \cup E'_2$ ;
- 21: Output  $\xi(E') = \{(n_b, n_w) \mid (b, w) \in E'\}$ , where  $(b \in BF(G)$  and  $w \in WF(G) \cup HF(G))$  and  $(n_b \in \rho^{-1}(b)$  and  $n_w \in \rho^{-1}(w) \cap V_W)$ .  $\square$

#### Procedure FindEdgesBP

**Input:** Leaf sets  $L_p$ ,  $B_p$  and  $W_p$ , and a linear ordering  $\beta_p$  on  $L_p = \{v_1, \dots, v_{|L_p|}\}$  with indices representing the order of  $\beta_p$ .

**Output:** An edge set  $E'_p$ .

- 1: **if**  $|L_p| = 2$  **then**
- 2:  $E'_p \leftarrow \{(v_1, v_2)\}$ , where  $v_1 \in W_p$  and  $v_2 \in B_p$  without loss of generality;
- 3: **else if** there exists a subscript  $a$  satisfying either  $(v_a \in B_p$  and  $v_{a+|L_p|/2} \in W_p)$  or  $(v_a \in W_p$  and  $v_{a+|L_p|/2} \in B_p)$  **then**
- 4: /\* Execute Steps 5 and 6 \*/
- 5: Regard  $v_c$  as  $v_1$  and assume that  $v_a \in W_p$  without loss of generality;

- Let  $L_{p_1} = \{v_2, \dots, v_{\lfloor L_p/2 \rfloor}\}$  and  $L_{p_2} = \{v_{2+\lfloor L_p/2 \rfloor}, \dots, v_{\lfloor L_p \rfloor}\}$ ;
- 6:  $E'_p \leftarrow \mathcal{M} \cup \{(v_1, v_{1+\lfloor L_p/2 \rfloor})\}$  (see Fig. 5), where  $\mathcal{M}$  is a matching with  $|\mathcal{M}| = \lfloor L_p/2 \rfloor - 1$  consisting of edges  $(u, v)$  satisfying  $(u \in L_{p_1} \cap W_p \text{ and } v \in L_{p_2} \cap B_p)$  or  $(u \in L_{p_1} \cap B_p \text{ and } v \in L_{p_2} \cap W_p)$ ; /\* Since  $|B_p| = |W_p|$  and  $|L_{p_1}| = |L_{p_2}|$ , such a matching  $\mathcal{M}$  exists. \*/
- 7: **else** /\* Such a pair as in Step 3 does not exist. \*/
- 8: /\* Execute Steps 9–11 \*/
- 9: Find any subscript  $c$  satisfying either  $(v_c \in W_p \text{ and } v_{c+1} \in B_p)$  or  $(v_{\lfloor L_p \rfloor} \in W_p \text{ and } v_1 \in B_p)$ ;
- 10: Regard  $v_c$  as  $v_1$  and assume that  $v_c \in W_p$  without loss of generality;  
Let  $L_{p_1} = \{v_3, \dots, v_{\lfloor L_p/2 \rfloor}\}$  and  $L_{p_2} = \{v_{3+\lfloor L_p/2 \rfloor}, \dots, v_{\lfloor L_p \rfloor}\}$ ;  
/\* Then  $v_{1+\lfloor L_p/2 \rfloor} \in W_p$  and  $v_{2+\lfloor L_p/2 \rfloor} \in B_p$  \*/
- 11:  $E'_p \leftarrow \mathcal{M}' \cup \{(v_1, v_{2+\lfloor L_p/2 \rfloor}), (v_2, v_{1+\lfloor L_p/2 \rfloor})\}$  (see Fig. 6), where  $\mathcal{M}'$  is a matching with  $|\mathcal{M}'| = \lfloor L_p/2 \rfloor - 2$  consisting of edges  $(u, v)$  satisfying  $(u \in L_{p_1} \cap W_p \text{ and } v \in L_{p_2} \cap B_p)$  or  $(u \in L_{p_1} \cap B_p \text{ and } v \in L_{p_2} \cap W_p)$ ;  
/\* Since  $|B_p| = |W_p|$  and the  $|L_{p_1}| = |L_{p_2}|$ , such a matching  $\mathcal{M}'$  exists. \*/
- 12: **end if**
- 13: Output  $E'_p$ . /\*  $E'_p$  is legal and  $|E'_p| = \lfloor L_p/2 \rfloor$  \*/  $\square$

### C. Correctness of the Algorithm

We prove correctness of the algorithm by using several lemmas. First, we have the next lemma for a structural graph  $F(G)$ .

**Lemma 4.1** [12] In  $F(G)$  with  $|LF(G)| \geq 4$ , if there are distinct four leaves  $v, w, x, y \in V(F(G))$  with  $\beta(v) < \beta(x) < \beta(w) < \beta(y)$  then there are distinct four vertices  $n_v, n_w, n_x, n_y \in V(G)$  such that  $|LF(G + \{n_v, n_w\})| = |LF(G)| - 2$  and  $|LF(G + \{n_x, n_y\})| = |LF(G)| - 2$ .

The lemma for the case with  $|LF(G)| = 4$  follows from Proposition 3.1 and Lemma 4.1.

**Lemma 4.2** [12] Assume that  $|LF(G)| = 4$ .

- (i) If  $|BF(G)| \geq 3$  then there exists an optimal solution  $E_f$  with  $|E_f| = |BF(G)|$ .
- (ii) If  $|BF(G)| \leq 2$  then let us consider a graph  $G_c$  defined in Proposition 3.1.
- (ii-1) If  $G_c$  is a simple cycle of length four such that two black leaves and two other ones that are either white or hybrid ones appear alternately (see Fig. 8) then there exists an optimal solution  $E_f$  with  $|E_f| = 3$ ;
- (ii-2) otherwise, there exists an optimal solution  $E_f$  with  $|E_f| = 2$ .

Next we are going to prove the next lemma on a subset of an optimal solution to 3ECABP for a modified cactus  $F(G)$  with  $\lambda(F(G)) = 2$  and  $|LF(G)| \neq 4$ .

**Lemma 4.3** For a structural graph  $F(G)$  with  $|LF(G)| \neq 4$ , **Procedure** FindEdgesBP finds an edge set  $E'_p$  to be added to  $F(G)$  such that  $L_p$  is included in a 3-component  $S$  of  $F(G) + E'_p$  and  $E'_p$  is legal.

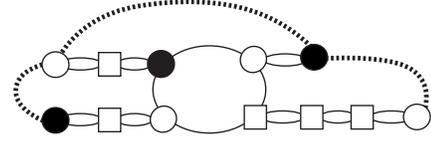


Fig. 8. Schematic explanation of Lemma 4.2 (ii-1), where the set of dashed lines is an optimal solution  $E'$  to a structural graph  $F(G)$ .

*Proof:* Clearly, for any  $(v, w) \in E'_p$ ,  $\{v, w\}$  is included in a 3-component of  $F(G) + E'_p$  because there are three edge-disjoint paths between  $v$  and  $w$  in the graph. Furthermore, it is well-known that  $\{u_1, u_2, u_3\}$  is a subset of a 3-component if there are three edge-disjoint paths between  $u_1$  and  $u_2$  and between  $u_2$  and  $u_3$ .

Now we consider the three cases:

- (i)  $|L_p| = 2$ ;
- (ii)  $|B_p| = |W_p| \geq 2$  and a subscript  $a$  satisfying either  $(v_a \in B_p \text{ and } v_{a+\lfloor L_p/2 \rfloor} \in W_p)$  or  $(v_a \in W_p \text{ and } v_{a+\lfloor L_p/2 \rfloor} \in B_p)$  exists;
- (iii)  $|B_p| = |W_p| \geq 3$  and any subscript  $a$  satisfying either  $(v_a \in B_p \text{ and } v_{a+\lfloor L_p/2 \rfloor} \in W_p)$  or  $(v_a \in W_p \text{ and } v_{a+\lfloor L_p/2 \rfloor} \in B_p)$  does not exist.

[The case (i)] Clearly the lemma follows.

[The case (ii)] We have  $\lambda(v_1, v_{1+\lfloor L_p/2 \rfloor}; F(G) + E'_p) \geq 3$ . Let  $u$  be any vertex in  $L_p - \{v_1, v_{1+\lfloor L_p/2 \rfloor}\}$ . Let  $(u, v) \in E'_p$ . If  $u \in L_{p_1}$  then  $w \in L_{p_2}$ , while if  $u \in L_{p_2}$  then  $w \in L_{p_1}$ . Since the discussion is symmetric for the two cases where  $u \in L_{p_1}$  or  $u \in L_{p_2}$ , we explain only the case with  $u \in L_{p_1}$ . As shown in Fig. 9,  $F(G) + E'_p$  has three edge-disjoint  $(v_1, u)$ -paths  $P_1, P_2$  and  $P_3$ . Hence  $L_p$  is included in a 3-component of  $F(G) + E'_p$ .

[The case (iii)] We have  $\lambda(v_i, v_{i+\lfloor L_p/2 \rfloor}; F(G) + E'_p) \geq 3$  for  $i = 1, 2$ . Let  $u$  be any vertex in  $L_p - \{v_1, v_2, v_{1+\lfloor L_p/2 \rfloor}, v_{2+\lfloor L_p/2 \rfloor}\}$ . Let  $(u, w) \in E'_p$ . If  $u \in L_{p_1}$  then  $w \in L_{p_2}$ , while if  $u \in L_{p_2}$  then  $w \in L_{p_1}$ . Since the discussion is symmetric, we explain only the case with  $u \in L_{p_1}$ . As shown in Fig. 10 (1),  $F(G) + E'_p$  has three edge-disjoint  $(v_1, v_2)$ -paths  $P_1, P_2$  and  $P_3$ . Furthermore, as in Fig. 10 (2), there are three edge-disjoint  $(v_1, u)$ -paths  $P'_1, P'_2$  and  $P'_3$  of  $F(G) + E'_p$ . Hence  $L_p$  is included in a 3-component of  $F(G) + E'_p$ .

Clearly  $E'_p$  is legal.  $\square$

Now the correctness of the algorithm is proved by the next lemma.

**Lemma 4.4** For a given graph  $G = (V, E)$  with  $\lambda(G) = \sigma$  and a bipartition  $\{V_B, V_W\}$  of  $V$ , **Algorithm** S-Sol $_{-}(\sigma + 1)$ ECABP finds an optimal solution  $E_f$ .

*Proof:* We prove the next lemma by showing that  $V(F(G))$  is only one 3-component in  $F(G) + E'$  and by counting  $|E'|$ , where  $E'$  is an optimal solution to 3ECABP for  $F(G)$ .

For any edge set found in Step 9, 14, 18 or 20, there are three cases as follows:

- (i)  $|LF(G)| = 4$ ;
- (ii)  $|LF(G)| \neq 4$  and  $F(G)$  is not  $B$ -dominant;
- (iii)  $|LF(G)| \neq 4$  and  $F(G)$  is  $B$ -dominant.

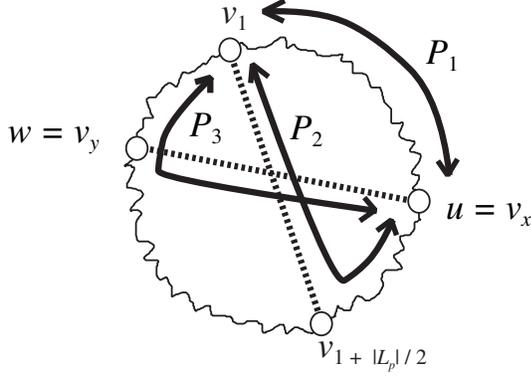


Fig. 9. Schematic explanation of three edge-disjoint  $(v_1, u)$ -paths of  $F(G)+E'_p$  in the case (ii), where an  $ET(F(G))$  is denoted by wavy lines.

[The case (i)] Lemma 4.2 shows that Step 9 finds an edge set  $E'$  with  $|E'| = \mathcal{L}$  (see Proposition 3.1 (i)) such that  $LF(G)$  is included in a 3-component of  $F(G) + E'$ .

[The case (ii)] We have  $B = \emptyset$  just before Step 17 of S-Sol $_{(\sigma+1)}$ ECABP. Lemma 4.3 shows that Steps 10–16 find an edge set  $E' = E'_1$  with  $|E'| = |E'_1| = |LF(G)|/2 = \mathcal{L}$  (see Proposition 3.1 (iii)) such that  $LF(G)$  is included in a 3-component of  $F(G) + E'$ .

[The case (iii)] Steps 10–16 of S-Sol $_{(\sigma+1)}$ ECABP find an edge set  $E'_1$  with  $|E'_1| = |W|$  and we have  $B \neq \emptyset$  just before Step 17. Since, for each  $(b, w) \in E'_2$ ,  $\{b, w\}$  is a subset of a 3-component of  $F(G) + E'_1$ , Step 20 gives us an edge set  $E' = E'_1 \cup E'_2$  with  $|E'| = |BF(G)| = \mathcal{L}$  (see Proposition 3.1(iii)) such that  $LF(G)$  is included in a 3-component of  $F(G) + E'$ .

Clearly  $E'_p$  is legal in all cases.

Suppose that  $F(G) + E'_p$  has a 2-cut  $(X, V(F(G)) - X; F(G))$  in the case (i), (ii) or (iii). Then it is a 2-cut of  $F(G)$  and, therefore, both  $X$  and  $V(F(G)) - X$  include at least one leaf of  $F(G)$ . However this is not possible, since all leaves of  $F(G)$  are included in a 3-component of  $F(G) + E'_p$ . Hence  $V(F(G))$  is a 3-component of  $F(G) + E'_p$  in all the cases (i), (ii) and (iii).  $\square$

#### D. Time Complexity

A structural graph  $F(G)$  can be constructed in  $O(|V||E| + |V|^2 \log |V|)$  time [10]. Since all  $(\sigma + 1)$ -components are extracted in linear time [14], [13], [8], [15] when  $1 \leq \sigma \leq 2$ , a structural graph can be constructed in linear time in this case. Step 5 can be done in  $O(|V|)$  time. FindEdegsBP and Step 18 can be done in  $O(|V|)$  time. Finally, an optimal solution  $E'$  to  $F(G)$  can be converted into an optimal one  $E_f$  to  $G$  in  $O(|E|)$  time.

From above discussion, Proposition 3.1 and Lemma 4.4, Theorem 1.1 follows.

#### V. CONCLUSION

In this paper, we have given a simplified  $O(|V||E| + |V|^2 \log |V|)$  time algorithm to find an optimal solution to

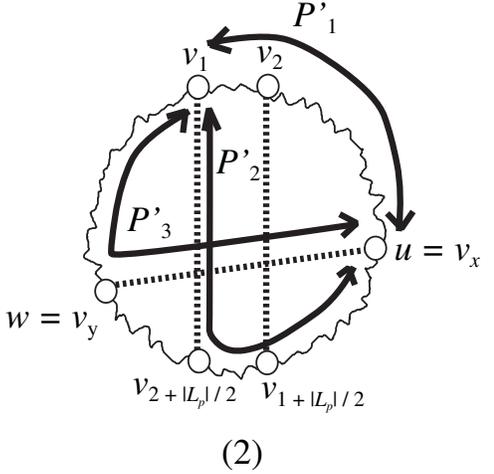
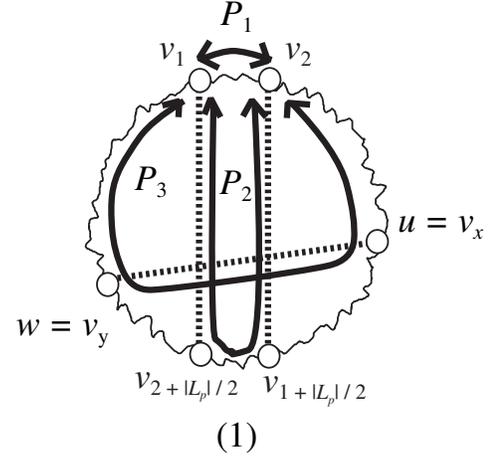


Fig. 10. Schematic explanation of (1) three edge-disjoint  $(v_1, v_2)$ -paths and (2) three edge-disjoint  $(v_1, u)$ -paths of  $F(G) + E'_p$  in the case (iii), where an  $ET(F(G))$  is denoted by wavy lines.

$(\sigma + 1)$ ECABP when  $\sigma = \lambda(G)$ . Moreover, it is shown that the problem can be solved in linear time when  $1 \leq \sigma \leq 2$ .

Note that, by means of S-Sol $_{(\sigma+1)}$ ECABP, we can easily solve a problem such that a subset  $\Gamma \subseteq V$  is additionally given in  $k$ ECABP and we require that  $\lambda(\Gamma; G) \geq (\sigma + 1)$  when  $\sigma = \lambda(V; G) = \lambda(\Gamma; G)$ .

Giving an efficient algorithm for  $(\sigma + \delta)$ ECAMP with  $\sigma = \lambda(G)$  and  $\delta > 1$  under several conditions is left as future research.

#### ACKNOWLEDGEMENTS

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